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INTRODUCTION TO DIMENSIONAL ANALYSIS AND THE THEORY OF NATURAL UNITS

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ABSTRACT:

This report presents a somewhat abbreviated introduction to dimensional analysis for students of science or engineering. It shows how to construct a system of consistent natural units appropriate to any given physical problem or context. It also explains how the well known Pi Theorem of dimensional analysis follows from this treatment, and how the dimensionless pi's of the theorem simply represent various physical quantities of interest as expressed in such natural units. The method is illustrated by application to the case of an ideal propeller. This example shows how dimensional analysis may be used to generalize and simplify a problem, and to extract the maximum degree of useful information and insight from its solution.

CONTENTS

1. Inertial-Mechanical Units

Several families or conventions of units are in common use in engineering. However, in order to explain the principles of dimensional analysis as simply as possible, it is advantageous to adhere consistently to one particular family. For this purpose we shall here utilize what we term the inertial-mechanical family, as explained further below. This choice represents what is perhaps the simplest and most common convention, at least in aeronautical engineering.

By an inertial system, we mean one in which the units of force, mass, length and time are so chosen that unit force imparts unit acceleration to unit mass. Under these conditions, Newton's law of motion takes the form

$$f = ma \tag{1-1}$$

where f is force, m is mass and a is acceleration.

Each quantity which appears in any physical equation such as Eq (1-1) is typically associated with some definite unit in some particular system of consistent units. Thus the force f in Eq. (1-1) is expressed in pounds if the English system is being used, or in newtons if the metric system is being used. In most cases it suffices simply to label such quantities with the names of their respective units at appropriate points in the course of any written computation.

For purposes of dimensional analysis, however, it is considerably more useful to represent every such unit by a corresponding generalized

symbol. Thus the chosen unit of force is generally represented by the letter F, the unit of mass by M, the unit of length by L and the unit of time by T. We term the symbols F, M, L, T generalized units or dimensions.

One advantage of this scheme is its flexibility. Thus, written relations involving these symbols remain valid whether the symbols are taken to represent English units or metric units or any other set of consistent units which conform to the inertial-mechanical convention.

In Eq. (1-1) force f, mass m, and acceleration a can therefore be said to have generalized units of force F, mass M, and acceleration $\frac{L}{T^2}$, respectively. Hence, by labelling each quantity in Eq. (1-1) with its appropriate generalized unit, we thereby obtain the following dimensional relation, namely,

$$F = \frac{ML}{T^2} \tag{1-2}$$

This expression can be regarded as defining the size of the consistent derived unit F in terms of the given magnitudes of the fundamental units M, L, and T. This means that if the sizes of the units corresponding to M, L, and T be specified arbitrarily, the size of the consistent inertial unit of force F is thereby fixed accordingly.

Eq. (1-2) can also be inverted to read

$$M = \frac{FT^2}{L} \tag{1-3}$$

This form,which is entirely equivalent to the preceding one, can be regarded as defining the size of the constant derived unit M in terms of the fundamental and arbitrary units F, L, and T.

A commonly used alternative to the inertial system of units is the gravitational system. In the gravitational system, unit force is defined as equal to the weight of unit mass in the earth's gravitational field, under certain specified standard conditions. For gravitational units, Newton's law takes the form

$$f = \frac{1}{g_o} ma \tag{1-4}$$

where g_o, the inertial constant, is numerically equal to the standard acceleration of gravity. Because of the presence of this factor, the units and dimensions F, M, L, T do not satisfy the inertial relation (1-2) or (1-3). It therefore simplifies the present discussion to exclude gravitational units from consideration.

By a mechanical system of units, we mean one in which work, heat and all other forms of energy are expressed consistently in mechanical units, and in which thermal units of energy are not used. The mechanical unit of energy is defined as equal to the work done by unit force on unit displacement. On the other hand, the thermal unit of energy is defined as equal to the heat required to raise the temperature of unit mass of water by unit temperature under certain specified standard conditions. For example, in the English system the mechanical unit of energy is the foot pound whereas the thermal unit would be the Btu. In the metric system the mechanical unit of energy is the newton meter or joule whereas the thermal unit would be the kilo-calorie. In any mechanical system, work, heat or energy in any form are all assigned the unit of energy E according to the relation for mechanical work, that is

$$E = FL \tag{1-5}$$

Thermal units of energy do not satisfy Eq. (1-5) and hence are excluded from this analysis.

Fortunately, the restriction to inertial-mechanical units is not a severe one because all quantities, however expressed originally, can always be converted to inertial-mechanical units without difficulty.

2. Fundamental and Derived Units

When using the inertial-mechanical convention for problems of mechanics, the generalized units or dimensions of all quantities can be expressed in terms of three fundamental dimensions, namely, either F, L, T, or M, L, T. If we wish to include the domain of thermodynamics, it becomes necessary to add the fundamental unit or dimension of temperature θ. To include the phenomena of electricity and magnetism, it would be necessary to add the fundamental dimension of electrical charge Q_E. In this discussion, however, electromagnetic effects are not considered, so that for our purposes the four necessary and sufficient fundamental dimensions are either F, L, T, θ, or M, L, T, θ.

The fundamental dimensions and the corresponding English and metric units in the inertial-mechanical convention are summarized in Table 2-1. The metric units shown are known as MKS units (meter, kilogram, second). An alternative, not shown, is the CGS system (centimeter, gram, second).

The generalized units or dimensions of all kinds of physical quantities can now be expressed in terms of the four chosen fundamental units. This fact is summarized in Table 2-2 which summarizes the generalized units of various typical derived quantities for any inertial-mechanical system of units. However, each quantity is shown first in terms of its

Table 2-1 Fundamental Units in the Inertial-Mechanical Family

Quantity	Symbol for Generalized Unit	English Units	Metric MKS Units
Force	F	pound	newton*
Mass	M	slug*	kilogram
Length	L	foot	meter
Time	T	second	second
Temperature	θ	degree Fahrenheit	degree Centigrade

Note: The asterisk * denotes the derived unit.

Table 2-2 Consistent Derived Units in the Inertial-Mechanical Family

Quantity	Conventional Generalized Units	Inertial-Mechanical Generalized Units		Inertial-Mechanical Standard Units	
	F,M,L,T,θ,E Basis	F,L,T,θ Basis	M,L,T,θ Basis	English	Metric MKS
Force	F	F	ML/T^2	lb	nwt = kg m/sec^2
Mass	M	FT^2/L	M	slug = lb sec^2/ft	kg
Angle	1	1	1	rad	rad
Length	L	L	L	ft	m
Area	L^2	L^2	L^2	ft^2	m^2
Volume	L^3	L^3	L^3	ft^3	m^3
Velocity	L/T	L/T	L/T	ft/sec	m/sec
Acceleration	L/T^2	L/T^2	L/T^2	ft/sec^2	m/sec^2
Pressure	F/L^2	F/L^2	M/LT^2	lb/ft^2	nwt/m^2
Energy	E	FL	ML^2/T^2	ft lb	joule = (m)(nwt)
Specific enthalpy	E/M	L^2/T^2	L^2/T^2	ft lb/slug = ft^2/sec^2	joule/kg = m^2/sec^2
Moment	FL	FL	ML^2/T^2	lb ft	(nwt)(m)
Momentum	ML/T	FL^2/T^3	ML/T	slug ft/sec = lb ft^2/sec^3	kg m/sec

Table 2-2 Consistent Derived Units in the Inertial-Mechanical Family (Continued)

Specific Impulse	FT/M	L/T	L/T	lb sec/slug = ft/sec	nwt sec/kg = m/sec
Power	FL/T	FL/T	ML^2/T^3	ft lb/sec	joule/sec = watt
Density	M/L^3	FT^2/L^4	M/L^3	slug/ft^3 = lb sec^2/ft^4	kg/m^3
Viscosity	FT/L^2	FT/L^2	M/LT	lb sec/ft^2	kg/ft sec
Specific Heat	$E/M\theta$	$L^2/T^2\theta$	$L^2/T^2\theta$	ft lb/slug $^\circ$F = ft^2/sec^2 $^\circ$F	joule/kg $^\circ$C
Specific Entropy	$E/M\theta$	$L^2/T^2\theta$	$L^2/T^2\theta$	ft lb/slug $^\circ$F = ft^2/sec^2 $^\circ$F	joule/kg $^\circ$C
Thermal Conductivity	$E/ML\theta$	$L/T^2\theta$	$L/T^2\theta$	lb/slug ft $^\circ$F = ft/sec^2 $^\circ$F	joule/kg m $^\circ$C

conventional definition which may involve any of the six generalized units, F, M, L, T, θ, E. Subsequently, energy E is eliminated by use of Eq. (1-5) and either F is eliminated by use of the inertial constraint Eq. (1-2) or M is eliminated by use of Eq. (1-3). The reduced inertial-mechanical units obtained in this way are shown in the two columns headed "F, L, T, θ basis" and "M, L, T, θ basis." These two dimensional representations are entirely equivalent. Either version may be used for purposes of dimensional analysis.

The inertial-mechanical units are further illustrated in Table 2-2 by two standard systems, namely, the English and metric MKS systems. Some of these units can be designated in more than one way as indicated, but in all such cases the alternative designations are entirely equivalent. In every case these units have precisely the dimensions indicated in the other columns, as discussed above. A few of the quantities tabulated, like specific heat for example, may have units that seem unfamiliar. The more familiar units for such quantities are often gravitational instead of inertial, and use thermal rather than mechanical units of energy. Such forms are not considered here. All quantities must be converted to the generalized units prescribed in Table 2-2 in order to fall within the scope of the present analysis. Fortunately, this conversion is always an easy matter.

Note that the generalized units shown in either the F, L, T, θ column or in the M, L, T, θ column, apply equally well not only to actual English and metric units, but also to all possible sets of inertial-mechanical units, including those hypothetical sets in which the magnitudes of all four fundamental units are specified arbitrarily.

A very important feature of these relationships must now be pointed out. This lies in the fact that <u>any physical equation which is valid for any one set of inertial-mechanical units remains valid and unchanged in mathematical form for every possible set of inertial-mechanical units!</u> In other words, the validity and mathematical form of every equation in the inertial-mechanical family remains unaffected by any change in the magnitudes assigned to the four fundamental units, provided only that all derived units are adjusted accordingly so as to maintain a consistent inertial-mechanical system, according to the dimensional relations shown in Table 2-2. Of course a change, say from English to metric units, affects the numerical values of all terms in any physical equation, but all terms in any one equation are changed by exactly the same numerical factor, so that the validity of the equation remains unaffected by the shift. The student can easily verify this by examining specific examples. In fact, this important principle is completely general. We shall be able to exploit it later to good advantage.

3. Natural Units

Suppose that in connection with any given problem or phenomenon, we can identify four physical parameters, call them A, B, C, D, as being of basic physical significance. On the basis of these four reference parameters, it is possible to construct a system of consistent inertial-mechanical units. Since these units are defined in terms of reference quantities which are intrinsic to the phenomenon under study, we call the units so obtained intrinsic or natural units.

Suppose that the four reference quantities are initially expressed in terms of fixed English or metric inertial-gravitational units. These units can always be expressed in the form

$$
\begin{aligned}
U_A &= F^{f_A} L^{\ell_A} T^{t_A} \theta^{\tau_A} \\
U_B &= F^{f_B} L^{\ell_B} T^{t_B} \theta^{\tau_B} \\
U_C &= F^{f_C} L^{\ell_C} T^{t_C} \theta^{\tau_C} \\
U_D &= F^{f_D} L^{\ell_D} T^{t_D} \theta^{\tau_D}
\end{aligned}
\qquad (3\text{-}1)
$$

where all exponents are known numbers, in accordance with Table 2-2.

On the basis of the four chosen reference parameters, it is now possible to define a new set of fundamental units which we designate as F^*, L^*, T^*, θ^*. These are defined by the expressions

$$
\begin{aligned}
F^* &= [AU_A]^{a_F}[BU_B]^{b_F}[CU_C]^{c_F}[DU_D]^{d_F} \\
L^* &= [AU_A]^{a_L}[BU_B]^{b_L}[CU_C]^{c_L}[DU_D]^{d_L} \\
T^* &= [AU_A]^{a_T}[BU_B]^{b_T}[CU_C]^{c_T}[DU_D]^{d_T} \\
\theta^* &= [AU_A]^{a_\theta}[BU_B]^{b_\theta}[CU_C]^{c_\theta}[DU_D]^{d_\theta}
\end{aligned}
\qquad (3\text{-}2)
$$

The exponents in Eqs. (3-2) can be determined uniquely in such a way that each of the four quantities F^*, L^*, T^*, θ^* will in fact have the correct dimensions. To show this, we rewrite the first of Eqs. (3-2) in the form

$$F^* = K_F F \qquad (3\text{-}3)$$

where

$$K_F = A^{a_F} B^{b_F} C^{c_F} D^{d_F} \tag{3-4}$$

$$F = U_A^{a_F} U_B^{b_F} U_C^{c_F} U_D^{d_F} \tag{3-5}$$

Now substituting Eqs (3-1) into (3-5) gives

$$\begin{aligned} F = &[F^{f_A} L^{\ell_A} T^{t_A} \theta^{\tau_A}]^{a_F} \times \ldots \\ &[F^{f_B} L^{\ell_B} T^{t_B} \theta^{\tau_B}]^{b_F} \times \ldots \\ &[F^{f_C} L^{\ell_C} T^{t_C} \theta^{\tau_C}]^{c_F} \times \ldots \\ &[F^{f_D} L^{\ell_D} T^{t_D} \theta^{\tau_D}]^{d_F} \end{aligned} \tag{3-6}$$

Regrouping exponents, we obtain

$$\begin{aligned} F = &F^{(f_A a_F + f_B b_F + f_C c_F + f_D d_F)} \times \ldots \\ &L^{(\ell_A a_F + \ell_B b_F + \ell_C c_F + \ell_D d_F)} \times \ldots \\ &T^{(t_A a_F + t_B b_F + t_C c_F + t_D d_F)} \times \ldots \\ &\theta^{(\tau_A a_F + \tau_B b_F + \tau_C c_F + t_D d_F)} \end{aligned} \tag{3-7}$$

Since exponents of like terms must be equal on both sides, this give

$$\begin{aligned} f_A a_F + f_B b_F + f_C c_F + f_D d_F &= 1 \\ \ell_A a_F + \ell_B b_F + \ell_C c_F + \ell_D d_F &= 0 \\ t_A a_F + t_B b_F + t_C c_F + t_D d_F &= 0 \\ \tau_A a_F + \tau_B b_F + \tau_C c_F + \tau_D d_F &= 0 \end{aligned} \tag{3-8}$$

Eqs. (3-8) may be solved for the four initially unknown exponents a_F, b_F, c_F, d_F as required in the first of Eqs. (3-2). Then Eq. (3-4) fixes K_F and Eq. (3-3) finally fixes F*, as required.

Proceeding in the same fashion for the second of Eqs. (3-2) we obtain

$$\begin{aligned} f_A a_L + f_B b_L + f_C c_L + f_D d_L &= 0 \\ \ell_A a_L + \ell_B b_L + \ell_C c_L + \ell_D d_L &= 1 \\ t_A a_L + t_B b_L + t_C c_L + t_D d_L &= 0 \\ \tau_A a_L + \tau_B b_L + \tau_C c_L + \tau_D d_L &= 0 \end{aligned} \qquad (3\text{-}9)$$

These equations may be solved for a_L, b_L, c_L, d_L whereupon

$$K_L = A^{a_L} B^{b_L} C^{c_L} D^{d_L} \qquad (3\text{-}10)$$

and

$$L^* = K_L L \qquad (3\text{-}11)$$

Applying the same method to the third of Eqs. (3-2) gives

$$\begin{aligned} f_A a_T + f_B b_T + f_C c_T + f_D d_T &= 0 \\ \ell_A a_T + \ell_B b_T + \ell_C c_T + \ell_D d_T &= 0 \\ t_A a_T + t_B b_T + t_C c_T + t_D d_T &= 1 \\ \tau_A a_T + \tau_B b_T + \tau_C c_T + \tau_D d_T &= 0 \end{aligned} \qquad (3\text{-}12)$$

whereupon

$$K_T = A^{a_T} B^{b_T} C^{c_T} D^{d_T} \qquad (3\text{-}13)$$

$$T^* = K_T T \qquad (3\text{-}14)$$

Finally from the fourth of Eqs. (3-2)

$$\begin{aligned} f_A a_\theta + f_B b_\theta + f_C c_\theta + f_D d_\theta &= 0 \\ \ell_A a_\theta + \ell_B b_\theta + \ell_C c_\theta + \ell_D d_\theta &= 0 \\ t_A a_\theta + t_B b_\theta + t_C c_\theta + t_D d_\theta &= 0 \\ \tau_A a_\theta + \tau_B b_\theta + \tau_C c_\theta + \tau_D d_\theta &= 1 \end{aligned} \qquad (3\text{-}15)$$

whereupon

$$K_\theta = A^{a_\theta} B^{b_\theta} C^{c_\theta} D^{d_\theta} \tag{3-16}$$

$$\theta^* = K_\theta \theta \tag{3-17}$$

The above four sets of simultaneous equations, namely, (3-8), (3-9), (3-12), and (3-15), can be solved, however, if and only if the following determinant is non-vanishing, that is

$$\begin{vmatrix} f_A & f_B & f_C & f_D \\ \ell_A & \ell_B & \ell_C & \ell_D \\ t_A & t_B & t_C & t_D \\ \tau_A & \tau_B & \tau_C & \tau_D \end{vmatrix} \neq 0 \tag{3-18}$$

Condition (3-18) amounts to a mild constraint on how the four reference parameters A, B, C, D may be chosen.

Suppose now that we have an arbitrary unit U_i in the fixed system. This unit is defined by the known values of the exponents in the expression

$$U_i = F^{f_i} L^{\ell_i} T^{t_i} \theta^{\tau_i} \tag{3-19}$$

The corresponding natural unit would be

$$U_i^* = F^{*f_i} L^{*\ell_i} T^{*t_i} \theta^{*\tau_i} \tag{3-20}$$

where the exponents are exactly the same numbers as in the preceding expression.

To analyze further the relation between the arbitrary fixed unit U_i and the corresponding natural unit U_i^*, we substitute into Eq. (3-20) the expressions (3-3), (3-11), (3-14), and (3-17). Then making use of (3-19) we obtain

$$U_i^* = \left[K_F^{f_i} K_L^{\ell_i} K_T^{t_i} K_\theta^{\tau_i} \right] U_i \tag{3-21}$$

This result confirms that U_i^* and U_i are indeed of like dimension, and differ only in magnitude, as required. It is convenient to rewrite Eq. (3-21) in the form

$$U_i^* = K_i U_i \tag{3-22}$$

where

$$K_i = K_F^{f_i} \; K_L^{\ell_i} \; K_T^{t_i} \; K_\theta^{\tau_i} \tag{3-23}$$

Now substituting into this relation the expressions (3-4), (3-10), (3-13) and (3-16) we obtain--

$$\begin{aligned} K_i = &\left[A^{a_F} B^{b_F} C^{c_F} D^{d_F}\right]^{f_i} \times \ldots \\ &\left[A^{a_L} B^{b_L} C^{c_L} D^{d_L}\right]^{\ell_i} \times \ldots \\ &\left[A^{a_T} B^{b_T} C^{c_T} D^{d_T}\right]^{t_i} \times \ldots \\ &\left[A^{a_\theta} B^{b_\theta} C^{c_\theta} D^{d_\theta}\right]^{\tau_i} \times \ldots \end{aligned} \tag{3-24}$$

Regrouping exponetial terms gives

$$\begin{aligned} K_i = &A^{(a_F f_i + a_L \ell_i + a_T t_i + a_\theta \tau_i)} \times \ldots \\ &B^{(b_F f_i + b_L \ell_i + b_T t_i + b_\theta \tau_i)} \times \ldots \\ &C^{(c_F f_i + c_L \ell_i + c_T t_i + c_\theta \tau_i)} \times \ldots \\ &D^{(d_F f_i + d_L \ell_i + d_T t_i + d_\theta \tau_i)} \end{aligned} \tag{3-25}$$

This result can finally be written in the more concise form

$$K_i = A^{a_i} B^{b_i} C^{c_i} D^{d_i} \tag{3-26}$$

where

$$
\begin{aligned}
a_i &= a_F f_i + a_L \ell_i + a_T t_i + a_\theta \tau_i \\
b_i &= b_F f_i + b_L \ell_i + b_T t_i + b_\theta \tau_i \\
c_i &= c_F f_i + c_L \ell_i + c_T t_i + c_\theta \tau_i \\
d_i &= d_F f_i + d_L \ell_i + d_T T_i + d_\theta \tau_i
\end{aligned}
\qquad (3\text{-}27)
$$

It is also possible to obtain a direct solution for exponents a_i, b_i, c_i, d_i as follows. We note firstly that the natural unit may be expressed in the form

$$U_i^* = [AU_A]^{a_i}[BU_B]^{b_i}[CU_C]^{c_i}[DU_D]^{d_i} \qquad (3\text{-}28)$$

This equation is then reduced in exactly the same manner as before. In this way we obtain the equations

$$
\begin{aligned}
f_A a_i + f_B b_i + f_C c_i + f_D d_i &= f_i \\
\ell_A a_i + \ell_B b_i + \ell_C c_i + \ell_D d_i &= \ell_i \\
t_A a_i + t_B b_i + t_C c_i + t_D d_i &= t_i \\
\tau_A a_i + \tau_B b_i + \tau_C c_i + \tau_D d_i &= \tau_i
\end{aligned}
\qquad (3\text{-}29)
$$

The solution of Eqs. (3-29) fixes the exponents a_i, b_i, c_i, d_i as required. The solution obtained is identical to that defined by Eqs. (3-27). If it is required to find the exponents a_i, b_i, c_i, d_i for more than four different units, solution by means of Eqs. (3-27) involves less work. For less than four units, Eqs. (3-29) entail less work.

Let X_i be the numerical value or measure of an arbitrary quantity when it is expressed in terms of the fixed unit U_i, and let X_i^* be the corresponding measure when the quantity is expressed in terms of the

corresponding natural unit U_i^*. Since it is the same physical quantity by either mode of description, we may therefore write

$$X_i U_i = X_i^* U_i^* \tag{3-30}$$

Also, by using (3-22) we obtain

$$X_i U_i = X_i^* U_i^* = X_i^* K_i U_i \tag{3-31}$$

Rearranging gives

$$X_i^* = \frac{X_i}{K_i} \tag{3-32}$$

Notice that the fixed unit U_i cancels identically from this result. Since X_i^* is now entirely independent of any fixed unit, it is dimensionless. Hence converting any dimensional quantity to natural units automatically converts it to dimensionless form, and makes its final value independent of the particular system of fixed units in terms of which all quantities happen to be expressed initially.

Now combining (3-26) with (3-32), we obtain finally

$$X_i^* = \frac{X_i}{A^{a_i} B^{b_i} C^{c_i} D^{d_i}} \tag{3-33}$$

where the exponents a_i, b_i, c_i, d_i can always be found from either Eqs. (3-27) or (3-29) such as to render X_i^* dimensionless. The denominator of (3-33) simply represents the size of the natural unit U_i^* as expressed in terms of the corresponding fixed unit U_i.

A dimensionless quantity of the above type is often represented by the Greek letter pi, that is, by the symbol Π_i instead of the symbol X_i^* as used in this discussion. It is therefore commonly known as a dimensionless pi. It is convenient to retain this terminology here despite the difference in our notation.

Notice that if the above non-dimensionalizing procedure be applied to the reference parameters A, B, C, D themselves, the result is always

$$A^* = B^* = C^* = D^* = 1 \tag{3-34}$$

An advantage of converting from fixed to natural units is that the magnitudes of fixed English or metric units are quite arbitrary and unrelated to the specific problem at hand, whereas the magnitudes of the natural units are always related in an appropriate way to the magnitudes of the reference parameters A, B, C, D which characterize the actual phenomena under study. In other words, natural units express all quantities on scales appropriate to the problem under consideration. Another advantage is that when expressed in natural units, the four reference parameters themselves assume unit magnitudes, thus in effect vanishing from the analysis. A third advantage is that all significant parameters take on a universal character; their numerical values become independent of the particular system of fixed units which happens to be used for expressing all quantities initially.

4. The Pi Theorem

Suppose that in any given physical situation, there exists one or more quantitative relationships, known or unknown, among a certain group of n physical quantities. Among these quantities, k fundamental dimensions are involved. For the present, we assume that the fundamental dimensions are specifically F, L, T, θ, so that $k = 4$.

We assume also that the above n quantities include the four dimensional reference parameters A, B, C, D. The question of just how these reference

parameters should actually be selected in any given case is considered later. For the present we consider them known. Let the remaining quantities be denoted by $X_1, X_2, \ldots X_i \ldots X_m$ where $m = n - k$.

Any relationship of the type mentioned above may be expressed symbolically in the form

$$f(A, B, C, D; X_1, X_2, \ldots X_i \ldots X_m) = 0 \tag{4-1}$$

Let us now convert all quantities to natural units. It has been shown that the generalized variable X_i in Eq. (4-1) is thereby reduced to the dimensionless form

$$X_i^* = \frac{X_i}{A^{a_i} B^{b_i} C^{c_i} D^{d_i}} \tag{4-2}$$

where the initially unknown exponents a_i, b_i, c_i, d_i can always be found by the methods explained in the previous section. For the sake of clarity, however, the essentials of the calculation are repeated below.

Assume that all the exponents in the following expressions are specified. Thus

$$\begin{aligned} U_A &= F^{f_A} L^{\ell_A} T^{t_A} \theta^{\tau_A} \\ U_B &= F^{f_B} L^{\ell_B} T^{t_B} \theta^{\tau_B} \\ U_C &= F^{f_C} L^{\ell_C} T^{t_C} \theta^{\tau_C} \\ U_D &= F^{f_D} L^{\ell_D} T^{t_D} \theta^{\tau_D} \end{aligned} \tag{4-3}$$

and

$$U_i = F^{f_i} L^{\ell_i} T^{t_i} \theta^{\tau_i} \tag{4-4}$$

The exponents required in Eq. (4-2) can now be found by solving the following set of equations, namely,

$$f_A a_i + f_B b_i + f_C c_i + f_D d_i = f_i$$

$$\ell_A a_i + \ell_B b_i + \ell_C c_i + \ell_D d_i = \ell_i$$

$$t_A a_i + t_B b_i + t_C c_i + t_D d_i = d_i \qquad (4\text{-}5)$$

$$\tau_A a_i + \tau_B b_i + \tau_C c_i + \tau_D d_i = \tau_i$$

An alternative method of solution for the exponents a_i, b_i, c_i, d_i is defined by Eqs. (3-27); it will not be repeated here.

It has earlier been noted that if the above non-dimensionalizing procedure be applied to the reference parameters A, B, C, D themselves, the results always reduce to

$$A^* = B^* = C^* = D^* = 1 \qquad (4\text{-}6)$$

It has been shown that any mathematical relation which applies among the original physical variables also applies unchanged if each physical variable X_i be replaced by its dimensionless counterpart, X_i^*. Hence Eq. (4-1) becomes

$$f(1, 1, 1, 1; X_1^*, X_2^*, \ldots X_i^* \ldots X_m^*) = 0 \qquad (4\text{-}7)$$

Thus the non-dimensionalizing procedure defined above reduces the number of quantities involved from n dimensional quantities to (n-k) dimensionless pi's. The statement that the original set of n dimensional quantities can always be reduced to an equivalent set of (n - k) dimensionless pi's in this way is known as the Pi Theorem. It was first stated by Buckingham, around 1925.

For problems in which temperature is not involved, the fundamental dimension θ may be dropped, whereupon k reduces from four to three. The

number of reference parameters required then also reduces to three, say, A, B, and C. Hence the fourth row and column may be deleted from Eqs. (4-3), and (4-5). Otherwise, the procedure is the same as before.

5. On Choosing Reference Parameters

In solving any given problem, the dimensional reference parameters A, B, C, D should be selected so far as possible from among the parameters that are--

1) highly significant.

2) independent or known.

3) relatively constant.

4) finite and non-vanishing.

Once a definite set of reference parameters has been selected, it is usually advisable to adhere to this set consistently throughout the entire course of a given problem. This guarantees that all results will be expressed on the basis of a single consistent and known set of natural units. A consistent approach of this kind represents a very powerful method of imposing the maximum possible degree of simplicity and order in the analysis of any complex physical phenomenon.

It is likewise desirable that every equation be expressed consistently in terms of the corresponding natural units, with each quantity expressed in its dimensionless version. When this is done, each relation is displayed in its more general and significant form, and all unessential aspects are eliminated. Moreover, all final numerical results then become wholly independent of the particular system of fixed units, whether English or metric, in which all dimensional quantities happen to be expressed initially.

6. An Example of the Method

As an example of the foregoing principles, consider the thrust f produced by an ideal propeller of disc area A when operating in a fluid of density ρ. The propeller is supplied with shaft power P. The relative forward velocity of the propeller with respect to the undisturbed fluid is V.

From the momentum theory of propellers, it is known that the foregoing parameters satisfy the relation.

$$f^3 = 2\rho AP(P - fV) \tag{6-1}$$

We are usually interested in the performance of a propeller of known size operating in a known medium and driven by an engine of known power. Hence ρ, A, P are obviously the appropriate reference parameters for this case. Only three parameters are needed since temperature θ is not involved in any of the five parameters of Eq. (6-1).

The dimensional exponents in the units corresponding to the three references parameters can now be summarized as follows.

$$\begin{aligned} U_\rho &= F^1 L^{-4} T^2 \\ U_A &= F^0 L^2 T^0 \\ U_P &= F^1 L^1 T^{-1} \end{aligned} \tag{6-2}$$

Let us first non-dimensionalize the thrust f. The dimensional exponents in the fixed unit of f are

$$U_f = U_\rho^a U_A^b U_P^c = F^1 L^0 T^0 \tag{6-3}$$

Subscripts are here omitted from exponents a, b, c for simplicity.

Substituting expressions (6-2) into (6-3) and equating exponents of like terms leads to three equations in the exponents of F, L, and T, respectively. These relations correspond to Eqs. (4-5). Thus

$$\begin{aligned} F:&\quad 1a + (0)b + 1c = 1 \\ L:&\quad -4a + 2b + 1c = 0 \\ T:&\quad +2a + (0)b - 1c = 0 \end{aligned} \tag{6-4}$$

The solution of these equations gives

$$\begin{aligned} a &= 1/3 \\ b &= 1/3 \\ c &= 2/3 \end{aligned} \tag{6-5}$$

The natural unit of force U_f^* is therefore related to the fixed unit U_f as follows.

$$U_f^* = \rho^{1/3}\, A^{1/3}\, P^{2/3}\, U_f \tag{6-6}$$

When expressed in this natural unit, the dimensionless thrust therefore becomes simply

$$f^* = \frac{f}{\rho^{1/3}\, A^{1/3}\, P^{2/3}} \tag{6-7}$$

Upon repeating this procedure for the other parameters, the following results are obtained. The symbol $\rightarrow$ means "is transformed to". Thus

$$\begin{aligned} \rho &\rightarrow \rho^* = 1 \\ A &\rightarrow A^* = 1 \\ P &\rightarrow P^* = 1 \\ f &\rightarrow f^* = \frac{f}{\rho^{1/3}\, A^{1/3}\, P^{2/3}} \\ V &\rightarrow V^* = \frac{V}{\rho^{-1/3}\, A^{-1/3}\, P^{1/3}} \end{aligned} \tag{6-8}$$

Next we replace the five dimensional quantities in Eq. (6-1) by their dimensionless counterparts as defined by Eqs (6-8). The results may be summarized in the form

$$f^{*3} = 2(1 - f^*V^*) \tag{6-9}$$

This result expresses the relation between the dimensionless thrust f^* and the dimensionless forward speed V^* for an ideal propeller. It can be shown that any real propeller can approach but never exceed the ideal performance defined by Eq. (6-9). Hence the result in this form is highly significant.

Notice that Eq. (6-9) is much simplier to grasp and far more informative than the original dimensional version (6-1). However, this transformation can only be made on the basis of the dimensionless parameters defined in Eq. (6-8).

Imagine an investigator acquainted with Eq. (6-1) but unfamilar with the principles of dimensional analysis. It is highly unlikely that he would intuitively hit upon the simple looking yet sophisticated form (6-9), or upon the unfamilar parameters defined in (6-8). Even if these parameters were actually pointed out to him, it is unlikely that he could grasp their relevance without further explanation. Yet any student armed with a grasp of the orderly method of dimensional analysis explained in this paper can derive these quantities by a straightforward procedure and readily interpret their physical significance.

7. Extensions of the Theory

The principles introduced in this paper can be extended to other families of consistent generalized units such as the gravitational family and the thermal family, and various other combinations. Also, the significance of the various exponential relations that characterize dimensional analysis can be analyzed far more deeply than we have attempted to do here. Owing to the fundamental character and the great usefulness of dimensional methods in engineering and in all quantitative sciences, these aspects, besides being interesting in themselves, should repay careful study by the serious student. A far more detailed and extensive development of these topics may be found in the following reference, which provides a suitable sequel to the present paper. It also contains a further bibliography on the the subject.

8. Reference

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13. ABSTRACT

This report presents a somewhat abbreviated introduction to dimensional analysis for students of science or engineering. It shows how to construct a system of consistent natural units appropriate to any given physical problem or context. It also explains how the well known Pi Theorem of dimensional analysis follows from this treatment, and how the dimensionless pi's of the theorem simply represent various physical quantities of interest as expressed in such natural units. The method is illustrated by application to the case of an ideal propeller. This example shows how dimensional analysis may be used to generalize and simplify a problem, and to extract the maximum degree of useful information and insight from its solution.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Dimensions						
Fundamental dimensions						
Dimensionless numbers						
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Dimensionless parameters						
Dimensionless pi's						
Pi Theorem						
English Units						
Metric Units						
MKS Units						
CGS Units						
Consistent Units						
Inertial Units						
Gravitational units						
Fixed units						
Standard units						
Natural units						
Intrinsic units						
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Unit and measure						
Dynamic similarity						
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Propeller parameters						
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Mathematical invariance						